PHYS 798C Fall 2025
Lecture 26 Summary

Prof. Steven Anlage

I. THE KOSTERLITZ-THOULESS (KT) PHASE TRANSITION FOR 2D SUPERCONDUCTORS

One does not have true long-range order (LRO) in 1 or 2 dimensions. The 3D BCS ground state is a coherent state of Cooper pairs that maintains phase coherence over effectively infinite distance. In lower dimensions, the phase coherence is less strong. This is quantified by the two-point correlation function for the GL order parameter. In 3D one has $\langle \psi^*(\vec{r})\psi(\vec{r}')\rangle \sim$ constant as $|\vec{R}| = |\vec{r} - \vec{r}'| \to \infty$. Above T_c one finds,
$\langle \psi^*(\vec{r})\psi(\vec{r}')\rangle \sim e^{-R/\xi_N}$, where $\xi_N \sim \xi_{GL} = \sqrt{\hbar^2/2m^*\alpha}$, with $\alpha > 0$ (see archived Former Lecture 26).
In 2D at low temperature one finds, $\langle \psi^*(\vec{r})\psi(\vec{r}')\rangle \sim \frac{1}{R^{\eta(T)}}$. This difference between the exponential decay above T_c to a power-law decay at low temperatures implies a phase transition. This is the famous "KT" transition that is seen in 2D Coulomb gases and in vortices in thin films of superfluid ^4He. The original reference is J. Phys. C 6, 181 (1973).

The theory examines the lowest energy excitations out of the ground state. Individual vortices have an energy that scales logarithmically with the size of the sample. Bound Vortex/Anti-vortex (V/AV) pairs have a much smaller energy, hence they dominate the low temperature properties. As temperature increases, there is an entropic advantage to de-pairing the V/AV pairs and creating un-bound free vortices. This precipitates the KT transition at T_{KT}. Below T_{KT} there is zero resistance in the limit of current going to zero. In other words the critical current is zero! Above T_{KT} there is finite resistance even in the limit as the current goes to zero.

II. KT IN 2D SUPERCONDUCTORS: ISOLATED VORTEX VS. V/AV ENERGIES

Vortices in 2D superconductors are similar to those discussed before in 3D superconductors except for the behavior of the magnetic field and current in the distant tails. Instead of having the currents falling off exponentially with distance for $r > \lambda$ in 3D, one instead has a surface current given by,

$$\mu_0 \vec{K}_s(r) = \frac{\Phi_0}{2\pi}\widehat{\theta} \times \begin{cases} \frac{d/\lambda^2}{r} & r << 2\lambda^2/d \\ \frac{2}{r^2} & r >> 2\lambda^2/d \end{cases}$$

See the paper by Pearl, Appl. Phys. Lett. 5, 65 (1964), posted on the class web site. The key things to note are the $1/r^2$ drop-off of the surface currents with distance, and the crossover length scale, called the perpendicular penetration depth $\lambda_\perp = 2\lambda^2/d$, where d is the film thickness. The crossover length scale can be macroscopic in size in low carrier density and/or disordered superconducting films of small (nm) thickness. The other case is a Josephson junction array, where the screening length is the Josephson penetration depth $\lambda_J \sim 1/\sqrt{J_c}$. By making the junction critical current density J_c small, the Josephson screening length can be in excess of many μm. Thus the $1/r$ "core" of the vortices can extend over macroscopic distances! The vortices now act like Coulomb charges interacting in a 2D metal, or like vortices in thin films of superfluid ^{4}He.

The energy of a free vortex can be calculated by ignoring the vortex core (GL $\kappa \to \infty$) and considering only the kinetic energy of the currents as,
$W_1 = \pi n^*_{s,2D} \frac{\hbar^2}{m^*} \ln \frac{R}{r_0}$, where $n^*_{s,2D} = n_s L$ is the 2D superfluid density, n_s is the 3D superfluid density, L is the length of the vortex (on the order of the film thickness), r_0 is the microscopic length scale where the current density approaches the de-pairing value (we expect $r_0 \sim \xi_{GL}$), and R is the sample size, where it is assumed that $\lambda_\perp$ is much greater than the sample size ($\lambda_\perp \gg R$). The energy of a single isolated vortex depends on the log of the ratio of the system size to a microscopic length scale, making it very expensive!

Contrast this with the case of a V/AV pair at some distance r apart. Far away from the pair (distances $>> r$) the flow fields of the two vortices cancel to good approximation, making the object appear "neutral" from far away. The currents are strong only within r, giving rise to a total energy of just,
$W_2 = 2\pi n^*_{s,2D} \frac{\hbar^2}{m^*} \ln \frac{r}{r_0}$.

Because $W_2 << W_1$ the V/AV excitations are the dominant excitations at low temperature in the extreme 2D superconductor.

The basic idea of KT-physics is that the elementary excitations out of the ground state are vortex/anti-vortex (V/AV) pairs. This will be the case in the limit of $\lambda_\perp >> R$, which is, admittedly, pretty exotic. Josephson junction arrays are the best way to get to this extreme 2D limit with superconductors. Your run of the mill Nb or Al thin films of thickness 10 nm, or so, are nowhere near this limit! In these cases, the Bogoliubons are the dominant excitations out the ground state.

The vortex/anti-vortex pairs are bound together at low temperatures and do not dissipate energy. The bound pairs can be dissociated by means of a strong transport current, giving rise to dissipation, and this process will be calculated later. Otherwise, an increase in temperature leads to the possibility of the thermal excitaiton of a single vortex. This process is the basis for the calcualtion of the Kosterlitz-Thouless transition temperature.

III. THE KOSTERLITZ-THOULESS PHASE TRANSITION FOR 2D SUPERCONDUCTORS

To naively estimate the KT transition temperature T_{KT}, calculate the Helmholtz free energy of a free vortex, $\Delta F_1 = W_1 - TS_1$ and see where it changes sign. The entropy S_1 comes from counting the number of microscopic configurations that give the same macroscopic properties. In the case of a free vortex added to the sample, the vortex could be located in any square of size a, where a is expected to be on the order of r_0. Thus the Helmholtz free energy can be written as,
$\Delta F_1 = W_1 - k_B T \ln\left(R^2/a^2\right)$.
This can be expanded as,
$\Delta F_1 = \left(\pi n^*_{s,2D}\frac{\hbar^2}{m^*} - 2k_BT\right)\ln\frac{R}{r_0} - 2k_BT\ln\frac{r_0}{a}$.
In the thermodynamic limit $R \to \infty$ only the first term survives. (Also we expect $\ln\frac{r_0}{a} \sim 0$ since both r_0 and a are on the scale of the vortex core size.)
Looking at the temperature where $\Delta F_1 = 0$ yields this implicit equation for T_{KT}:
$n^*_{s,2D}(T_{KT}) = \frac{2m^*k_B}{\pi\hbar^2}T_{KT}$. One can find T_{KT} by finding the intersection of $n^*_{s,2D}(T)$ and the line described by $\frac{2m^*k_B}{\pi\hbar^2}T$. The class web site shows such data from superfluid ^4He and In/InO_x superconducting films. From those plots one can see that the superfluid density is heading to zero at some higher mean field transition temperature, T_{c0}, but the V/AV fluctuations, and their un-binding, interrupts this mean field transition at a lower temperature T_{KT}. Hence this is a fluctuation-dominated phase transition.

IV. HIGHLIGHTS OF KT PHYSICS IN 2D SUPERCONDUCTORS

For temperatures above T_{KT} one can define a free-vortex correlation length $\xi_+(T) \sim r_0 e^{\sqrt{B\frac{T_{KT}}{T-T_{KT}}}}$, where B is a constant of order unity. For length scales less than $\xi_+(T)$ there are no free vortices. Hence $\xi_+^2(T)$ is a measure of the puddle size of free-vortex-free regions. Note that this length scale diverges as T_{KT} is approached from above. One can use it to estimate the free vortex density as $n_f(T) = 1/\xi_+^2(T)$, for $T > T_{KT}$. The free vortex density thus goes to zero at T_{KT}. The free vortices will dissipate energy when acted upon by an external current, thus T_{KT} can be found from the zero-resistance state of the material, in principle. An estimate of the resistivity of the sample is made in analogy with the Bardeen-Stephen law used in Lecture 20 (the total resistivity is the normal state resistivity times the fractional area coverage of vortex cores): $\rho(T) = \rho_n\frac{r_0^2}{\xi_+^2(T)}$, where ρ_n is the normal state resistivity of the film. This gives a very distinctive prediction for the resistivity temperature dependence above T_{KT}.

Now think about what happens below T_{KT} in the presence of a finite current. When a transport current is applied to a bound V/AV pair, the Lorentz force will act in opposite directions on each vortex and act to stretch the pair. This gives rise to a peak in the energy of the V/AV pair as a function of separation r. The V/AV pair can unbind due to a thermal fluctuation activating the system over the barrier, creating free vortices below T_{KT}. The free-vortex generation rate is given by
$G = G_0 e^{-E_0/k_BT}$, where $E_0 = q^2\ln\left(\frac{q^2}{r_0\Phi_0 j_{2D}}\right)$ is the height of the energy barrier, G_0 is the attempt frequency for jumping over the barrier, $q^2(T) = \frac{2\pi\hbar^2 n^*_{s,2D}(T)}{m^*}$, and $j_{2D} = LJ_s$ is the 2D surface current density. With these definitions, the free vortex generation rate is,

$G = G_0 \left(\frac{r_0 \Phi_0 j_{2D}}{q^2}\right)^{q^2/k_B T}$.
But free vortices generated this way can also re-combine and annihilate. This recombination rate is given by $R = R_0 n_f^2$, where n_f is the free vortex density induced by the finite current below T_{KT}.
By assuming a dynamic equilibrium and equating the generation and recombination rates, we can calculate the free vortex density as,
$n_f = \sqrt{\frac{G_0}{R_0}} \left(\frac{r_0 \Phi_0 j_{2D}}{q^2}\right)^{q^2/2k_B T}$, for $T < T_{KT}$ in the presence of a current.

Assuming $\rho \sim E/j_{2D} \sim n_f$, then these free vortices will create a longitudinal electric field given by, $E \sim j_{2D}^{a(T)}$ with $a(T) = 1 + \frac{\pi \hbar^2 n^*_{s,2D}(T)}{m^* k_B T}$. This exponent has the value of 3 at T_{KT}, and a value of 1 above T_{KT} (Ohmic dissipation due to free vortices, calculated above). More generally, the $E - j_{2D}$ relation can be written as $E \sim j_{2D}^{1+2\frac{T_{KT}}{T}\frac{n^*_{s,2D}(T)}{n^*_{s,2D}(T_{KT})}}$. This form shows that the exponent grows, starting from a value of 3, for $T < T_{KT}$. Thus the IV curves show a discontinuous jump in slope from 1 to 3 at T_{KT}, followed by a steady rise in slope below that temperature. The large value of the exponent at low temperature resembles a finite critical current.

Along with this there is a discontinuous drop to zero in superfluid density $n^*_{s,2D}$ at T_{KT}. This can be seen from the fact that the $E - j_{2D}$ exponent is $a(T) = 1 + \frac{\pi \hbar^2 n^*_{s,2D}(T)}{m^* k_B T}$, which must equal 1 when $T > T_{KT}$, hence this requires that $n^*_{s,2D} = 0$ above T_{KT}. The abrupt drop in superfluid/super-electron density at T_{KT} is a universal property of the KT transition.